

# ON THE CONVENTIONALITY OF SIMULTANEITY

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Starting from the experimental fact that light propagates over a closed path at speed  $c$  ( $L/c$  law), we show to what extent the isotropy of the speed of light can be considered a matter of convention. We prove the consistence of anisotropic and inhomogeneous conventions, limiting the allowed possibilities. All conventions lead to the same physical theory even if its formulation can change in form. The mathematics involved is that of gauge theories and the choice of a simultaneity convention is interpreted as a choice of the gauge. Moreover, we prove that a Euclidean space where the  $L/c$  law holds, gives rise to a spacetime with Minkowskian causal structure, and we exploit the consequences for the causal approach to the conventionality of simultaneity.

Key words: special relativity, conventionality, simultaneity, clock synchronization, Sagnac effect.

## 1 INTRODUCTION

Since its birth [1, 2], there has been a long debate to establish to what extent special relativity, and the hypothesis of constancy of the speed of light, could be considered conventional [3, 4]. It was soon realized by Einstein [2] that experimentally one can establish the speed of light only by measuring the time of flight of a light beam over a closed path. Indeed, in order to measure the *one-way* speed of light, the time of departure from a starting point  $O_1$  and the time of arrival to an ending point  $O_2$  are needed. A convention to

synchronize two distant clocks must be given, but the Einstein convention cannot be used, since it is based on the isotropic assumption which is the fact we wish to prove.

The situation is often illustrated, in the one dimensional case, in the following way [3](see figure 1). Let us consider a light beam: it leaves the

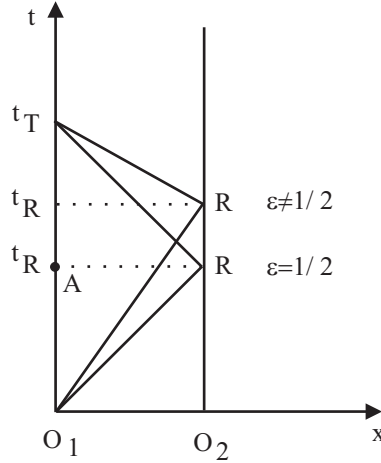


Figure 1: Different values of  $\epsilon$  give different definitions of simultaneity.

observer  $O_1$ , reaches  $O_2$  and, being reflected, it comes back.  $O_1$ , with his clock, measures the total time of flight,  $t_T$ , and verifies the relation  $2\overline{O_1O_2} = ct_T$ . If the speed of light is the same in both directions, the beam is reflected by  $O_2$  at the time  $t_R = t_T/2$ . This data, once communicated from  $O_1$  to  $O_2$ , can be used by  $O_2$  to synchronize his clock with  $O_1$ 's (the Einstein procedure of synchronization). In other words the assumption of isotropy leads to the conclusion that the events A and R are simultaneous, so that the definition of simultaneity suffers from the same conventionality content of the isotropic assumption.

It is often noticed by some authors that this conclusion cannot be drawn essentially for two reasons.

- There may be some way to synchronize distant clocks without using the isotropic assumption, for instance, with a slowly transport of a third clock from  $O_1$  to  $O_2$ .
- In the above argument we used only one experimental fact, that the speed of light as measured over a closed path is always  $c$ , (hereafter this law<sup>1</sup> is referred to, in short, as the “ $L/c$  law”). Other experimental facts

<sup>1</sup>Elsewhere [8] it is called “Weyl’s *Erfahrungstatsache*”, and it is distinguished from the

could restrict the allowed values of the speed of light in one direction, eventually leaving us with only the isotropic possibility.

However, the solution suggested in the first point simply replaces the isotropic convention with other equivalent assumptions [5, 6, 8]. It does not exclude the possibility of alternative anisotropic choices. As we shall see, the second argument does not work as well, because there are anisotropic conventions compatible with every physical law, the only price to be paid being a change in their mathematical expression. Moreover, conventions different from the isotropic one can prove to be natural in some contexts such as when the observers live over the surface of a spinning planet (Section 6).

As a first example of an alternative convention let us return to the one dimensional case. Following the supporters of the conventionality of simultaneity, we are able to fix the time reflection,  $t_R = \epsilon t_T$ , where  $\epsilon$  is the Reichenbach coefficient usually taken in the range  $\epsilon \in (0, 1)$ . Once the choice has been made, the speed of light directed right becomes  $c/2\epsilon$  and the one directed left becomes  $c/2(1 - \epsilon)$ . Whatever the choice of  $\epsilon$  is, the  $L/c$  law is satisfied. The restriction to the one dimensional case however does not clarify the problem, nor exhibits the richness of the possible conventions. Our analysis starts in the following section where we skip to the three dimensional case.

## 2 ANISOTROPY AND INHOMOGENEITY

Let us consider a Euclidean space  $E^3$  where light propagates on straight lines. A beam of light leaves its starting point  $O_1$  and through the reflection over suitable mirrors covers a closed path  $\gamma$  ending in  $O_1$ . If we use a large number of mirrors the path can approximate, as much as we want, a smooth closed curve of arbitrary shape, so that we can assume  $\gamma$  to be an arbitrary differentiable closed curve. If  $L$  is the length of the curve, by the  $L/c$  law, the total travelling time is  $\tau = L/c$ . Let us introduce a field  $\vec{A}(\vec{x})$  so that  $\nabla \times \vec{A} = 0$  (or, which is the same,  $\vec{A} = \nabla \phi$  for a suitable scalar function  $\phi(\vec{x})$ ), then

$$\tau = L/c + \oint_{\gamma} \vec{A} \cdot d\vec{l}. \quad (1)$$

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Reichenbach round-trip axiom which states the independence of the round-trip time from the direction of the journey.

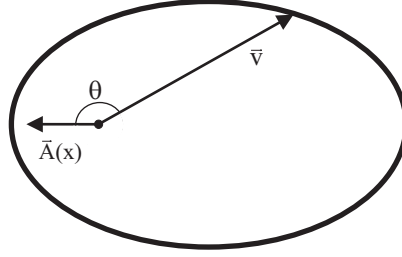


Figure 2: The anisotropy of the speed of light at the point  $\vec{x}$  is elliptical.

The previous expression can be rewritten

$$\tau = \oint_{\gamma} dl \left( \frac{1}{c} + A \cos \theta \right) = \oint_{\gamma} \frac{dl}{v(\vec{x}, \theta)}, \quad (2)$$

where  $\theta$  is the angle between  $\vec{A}$  and  $d\vec{l}$  and where

$$\vec{v} = \frac{c\hat{v}}{1 + c\hat{v} \cdot \nabla\phi(\vec{x})} \quad (3)$$

is a new modified speed of light. It is anisotropic, in fact its absolute value depends on the direction  $\hat{v}$  (see figure 2). Now it is clear that, if the speed of light has the anisotropic and inhomogeneous value given by Eq. (3), then the  $L/c$  law is fulfilled. Therefore, the arbitrariness of the speed of light amounts at least to an entire field  $\phi(\vec{x})$ .

There are more general expressions which verify the  $L/c$  law for any closed path starting at  $O_1$ . However, if the  $L/c$  law is verified by any observer, that is for any choice of  $O_1$ , then the most general expression for the velocity is given by Eq. (3). This is proved in the appendix, and is correctly stated by the following

**Theorem.** Let  $M = E^3 \times \mathbb{R}$  be a spacetime consisting of a Euclidean space  $E^3$  endowed with a global time  $\mathbb{R}$ . Suppose that any observer at rest measures with his clock, and along his worldline, a time which differs from the global time  $t$  only by an additive constant. Suppose moreover that light propagates along straight lines with a finite speed of norm  $f$ ,

$$\vec{v} = \frac{d\vec{x}}{dt} = f(\hat{v}, \vec{x}, t) \hat{v}. \quad (4)$$

Finally, assume that two light worldlines coincide if they pass through the same event with the same direction. Then, if the  $L/c$  law holds:

- a) In  $M$  there is a global time variable  $\eta$  that makes the speed of light isotropic

$$\vec{w} = \frac{d\vec{x}}{d\eta} = c\hat{w}. \quad (5)$$

The function  $\eta(\vec{x}, t)$  is unique up to an additive constant, and of the form

$$\eta = t - \phi(\vec{x}) \quad (6)$$

- b) If  $f$  is continuous in its arguments, the function  $\phi(\vec{x})$  is differentiable, with  $|\nabla\phi(\vec{x})| < 1/c$ , and the speed of light is given by Eq. (3).

Moreover, if light is, for any given direction, the fastest signal<sup>2</sup> that carries information then:

- c)  $M$  has a Minkowskian causal structure.

In the proof we construct, using the Einstein synchronization convention, a new global time variable  $\eta$  which makes the speed of light isotropic. Then we prove that  $\eta$ , like  $t$ , measures the time of clocks at rest (Eq. 6). The other statements follow a).

We notice that the first part of the theorem (firsts two points), applies to whatever signal propagates over straight lines such as, for example, the sound. In that case, with  $c$  we mean the *two-way* velocity of the signal.

Statement a) of the theorem is in some respect opposite to statement b). The latter implies the conventionality of the isotropic assumption whereas the former shows that all the freedom in choosing the convention can be eliminated with a time coordinate change. This last circumstance makes the one to one correspondence between the convention in the velocity distribution and the convention in the concept of simultaneity, clear. Two different time variables  $\eta$  and  $t$ , related by Eq. (6), have indeed, different simultaneity slices.

The theorem, moreover, suggests to replace the postulate of constancy of the speed of light with the  $L/c$  law which is independent from the synchronization convention used.

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<sup>2</sup> Experimentally it is possible, for any given direction, to establish unambiguously which of two signals is the fastest. Let the two signals start at the same instant from  $O_1$  and reach  $O_2$ . Clearly, the first of the two signal that reach  $O_2$ , as measured by  $O_2$ 's clock, is the fastest. This conclusion does not depend on the synchronization of the clocks of  $O_1$  and  $O_2$ .

Anderson and Stedman [7], starting from the isotropic convention and making the substitution (6), obtained the conventions expressed by (3). We have shown that even the converse is true: every convention for the velocity of light, allowed by the  $L/c$  law, takes that form and, hence, every allowed convention can be seen as arising from the isotropic convention by a change in the clocks synchronization. A step in this direction was already made by Anderson et al. [8, p. 132] who proved that such a coordinate change exists, provided the  $L/c$  law holds, and the anisotropy is elliptical [8, p. 127]. However, the strong requirement of ellipticity was made *a priori*, so that a proof of our theorem was still lacking.

We have derived the causal structure of  $M$  without the need of a spacetime metric. Let, now,  $M$  be a Lorentzian manifold where light rays propagate over null geodesics. By the theorem, its metric  $ds^2$ , in the coordinates  $\{\vec{x}, \eta\}$ , must equal Minkowski's up to a conformal factor (see, for instance, [11]). If additionally, the metric is required to measure the proper time of clocks at rest ( $ds = d\eta$ ), then the conformal factor is fixed to unity and the Minkowski metric is completely recovered.

Finally, the reader can verify, following the proofs of the appendix, that the conditions of the theorem can be considerably weakened allowing a generalization of the first part. The space  $E^3$  may be replaced with a Riemmanian space of  $\mathbb{R}^3$  topology where light is no longer required to propagate along the geodesics of the space, but needs only to move in trajectories invariant under inversion of direction. Then in Eq. (3) with  $\vec{v}$  we mean  $dx^i/dt$  and with  $\hat{v}$  we mean  $dx^i/dl$  where  $dl^2 = h_{ij}dx^i dx^j$  is the metric of the Riemmanian space. Here, we are mainly concerned with the theoretical and experimental foundations of Minkowski spacetime, so as not to discuss this generalization further.

### 3 CONSISTENCE OF ANISOTROPIC CONVENTIONS

We have seen that any possible convention is related to the isotropic one by the coordinate transformation of Eq. (6). This is the fundamental ingredient which allows us to prove our inability to find some physical phenomena ruling out one convention instead of another, as shown in great detail by Anderson et al. [8]. Let us shortly review their findings; this will justify the use of the word “convention” which, in the present paper, is referred to any scheme allowed by experience.

We can express all the known laws of physics in a conventional global time

coordinate obtaining a set of physical laws coherent with experience. The set is the one we had developed if, in our history of science, instead of using the isotropic convention we had chosen an anisotropic one. This set is recovered simply by performing a coordinate transformation from the coordinates of the isotropic convention,  $\{\vec{x}, \eta\}$ , to the coordinates of an anisotropic convention,  $\{\vec{x}, t\}$ . For instance, the Gauss law of electromagnetism is written in the anisotropic coordinates

$$\nabla \cdot \vec{E} + \nabla \phi \cdot \frac{\partial \vec{E}}{\partial t} = 4\pi\rho. \quad (7)$$

As another example, the velocity of a particle of worldline  $\vec{x}(t)$  is given by

$$\vec{v} = \frac{d\vec{x}}{dt} = \frac{\vec{w}}{1 + \vec{w} \cdot \nabla \phi}, \quad \text{where } \vec{w} = \frac{d\vec{x}}{d\eta}, \quad (8)$$

and the proper time of the particle is

$$d\tau = \sqrt{1 - \frac{\vec{w}^2}{c^2}} d\eta = \sqrt{(1 - \vec{v} \cdot \nabla \phi)^2 - \frac{\vec{v}^2}{c^2}} dt. \quad (9)$$

Not every physical law requires a time variable for its formulation. When it is possible, a convention-free formulation clearly reveals the physical meaning of the law. We have already shown that the causal structure is independent from the chosen convention. In  $M = E^3 \times \mathbb{R}$  we can even define, in a convention-free manner, the “light cone” of an event  $A$ ,

*An event  $B$  is said inside the light cone of an event  $A$  if there is a path  $\gamma$  which allows a light beam starting in  $A$ , and travelling over  $\gamma$ , to end in  $B$ .*

As a consequence, the law which states that information cannot propagate faster than light, can be expressed in a convention-free way, too.

In the literature there are theoretical proofs [9] of the isotropy of the speed of light but all of them use a non-modified law of physics in the process, and no one deals with the problem of velocity inhomogeneity. It is an easy task to “prove” the isotropy of the speed of light if we implicitly use an assumption or a law that holds only in the isotropic convention. For instance, it is easy to prove the isotropy of the speed of light if we use the Gauss law in its standard form. Indeed, its predictions agree with physical phenomena, Eq. (7), only if the velocity of light is isotropic. In the same way, contrary to the claims of Will [10], it is impossible to experimentally test the isotropy of the speed of light. Many of the experiments he had analyzed have some value as tests of special relativity, but as tests of the isotropy of the speed of light, they are illusions [8, p. 148].

## 4 SIMULTANEITY FROM CAUSALITY

In the previous section, we have shown that the laws of physics are simplified in the Einstein convention whereas, in anisotropic conventions, they lose their symmetries. The requirement of some symmetry becomes a way to restrict the allowed conventions to the isotropic one. This is seen even in the causal approach to the conventionality of simultaneity [4] whose cornerstone is the theorem of Malament [12]. This theorem (see [14] for a readable account of it) essentially proves that, if the causal structure of spacetime is that of Minkowski, and  $C$  is the worldline of an observer at rest, the only non trivial equivalence relation (simultaneity relation) invariant under  $C$ -causal automorphisms (diffeomorphisms of spacetime that preserve causal relations and map  $C$  onto itself) is that of Einstein. Our theorem enforces that of Malament in the following sense. The causal structure of spacetime is non-conventional because it can be tested experimentally. However, Malament takes it for granted that it is Minkowskian, that is, derived from a pseudo-Riemannian manifold of  $\mathbb{R}^4$  topology, where the metric

$$ds^2 = c^2 dt^2 - d\vec{x}^2 \quad (10)$$

vanishes on light worldlines. Now, one may wonder if this belief is well founded. After all, from (10), there also follows that the speed of light is isotropic (by dividing by  $dt^2$ ). In other words we cannot rely on equation (10) to state the causal structure of spacetime because it is only compatible with the isotropic assumption. We need experimental evidence that the causal structure of spacetime is Minkowskian and to do this we cannot rely on speed of light conventions. Alternatively, we need a proof of the independence of the causal structure from the convention chosen. At this point, the last part of our theorem enters. It states that, because of the  $L/c$  law, the causal structure of spacetime is Minkowskian even if our spacetime  $M = E^3 \times \mathbb{R}$  is not a Lorentzian manifold<sup>3</sup>. Malament's argument then works. We summarize the entire deduction in a scheme

$M = E^3 \times \mathbb{R} \xrightarrow{L/c \text{ law}} \begin{array}{c} \text{Minkowskian} \\ \text{causal structure} \end{array} \xrightarrow{\text{Malament's argument}} \text{Einstein convention.}$
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Although attractive, in what follows we abandon this approach to the conventionality of simultaneity essentially for one reason. Malament, in order

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<sup>3</sup>Remember that the causal structure of spacetime follows our theorem without the need of a spacetime metric. Moreover, Malament's argument uses only the causal structure of Minkowski spacetime.



to recover the Einstein convention, requires an invariance principle, namely the invariance of the simultaneity relation under C-automorphisms; but we have seen that a number of physical laws have the same effect if we require some symmetry. There is no physical reason for such a requirement; after all, the concept of simultaneity has to do with clocks not with light (causal structure) and in this regard clocks say that there are a number of viable conventions, those given by Eq. (3). Moreover, Malament's argument is hard to generalize to the case of observers in generic motion [13], or to generic spacetimes, because in such contexts C-automorphisms may be absent.

## 5 THE CHOICE OF A GOOD CONVENTION

Once one agrees on the conventional nature of simultaneity, the problem becomes how to find a good convention for the physical context at hand. We suggest three criteria

- Simplicity of the laws of physics.
- Invariance of the convention used under change of the observer.
- Existence of a global time variable.

The first criterion is clear, one has to choose the simplest convention whenever the last two points are satisfied. The second criterion has the following meaning: a convention is good if it is the same for every observer, in such a way that communication among them is possible without referring each time to a subjective choice. This implies that the function  $\phi$  must be the same for all observers: indeed, if the observer  $O_1$  uses the time variable  $t_1 = \eta_1 - \phi_1$  and the observer  $O_2$  uses the time variable  $t_2 = \eta_2 - \phi_2$ , a communication among them is useless, unless each observer knows the function  $\phi$  used by the other. To meet the second point, the set of observers must agree that function  $\phi$  be used. Depending on the physical situation, we have to restrict and define the set of observers under which the invariance of the convention holds. This happens in the following example.

So far we have considered only observers at rest, here we look for conventions well suited for moving observers. In this example, our set of observers, under which invariance of the convention must be met, is given by inertial observers. Let us consider the Galilean principle of relativity

- (\*) *A reference frame in uniform rectilinear motion with respect to an inertial frame is inertial by itself.*

Here, for “inertial frame”, we mean any observer who does not feel inertial forces. This definition, based on detectable forces, avoids any convention and is ideal for our purpose. The function  $\phi$ , common to all inertial observers, is taken in such a way that the Galilean relativity principle (\*) remains unchanged passing from the time variable  $\eta$  to the time variable  $t$ . This restricts the allowed conventions to a subset where the relativity principle holds and where the laws of physics happen to be particularly simple. Let  $\vec{w}$  be the uniform velocity of an inertial observer, from Eq. (8), we see that in the time variable  $t$  the inertial observer has a uniform velocity only if  $\nabla_{\vec{w}}\nabla_{\vec{w}}\phi = 0$  which implies<sup>4</sup>  $\phi = \vec{a}\cdot\vec{x} + \text{const.}$ . With this choice, the coordinate transformations from one inertial observer to another form a group. If  $U(a)$  is the coordinate transformation from  $\{\vec{x}, t\}$  to  $\{\vec{x}, \eta\}$  then the coordinate transformation to a second observer of velocity  $\vec{v}$  is given by [8, 17]

$$G(\vec{a}, \vec{v}) = U(a)^{-1}\Lambda(\vec{w})U(a) \quad (11)$$

where  $\vec{v}$  and  $\vec{w}$  are related by Eq. (8). The group of coordinate transformations can be shortly written  $G = U(a)^{-1}\Lambda U(a)$  where  $\Lambda$  is the Lorentz group in the usual representation. We take,  $\vec{a} = \frac{\alpha}{c}\hat{i}$ , where  $\alpha$  is a dimensionless constant. In one spatial dimension the modified Lorentz transformation becomes

$$x' = \gamma(w(v))\{(1 + \alpha\beta)x - \beta ct\} \quad (12)$$

$$ct' = \gamma(w(v))\{(1 - \alpha)\beta ct + (1 + \alpha - \alpha\beta + \alpha^2\beta)x\}, \quad (13)$$

where

$$\beta = \frac{w}{c} = \frac{v/c}{1 - \alpha v/c}, \quad (14)$$

$$\gamma(w(v)) = \frac{1 - \alpha v/c}{\sqrt{(1 - \alpha v/c)^2 - (v/c)^2}}. \quad (15)$$

With the simplest choice,  $\alpha = 0$ , we recover the Lorentz transformation.

We mention another interesting convention. Starting from a realistic viewpoint, and with purposes very different from ours [19]-[21], Selleri [18] renewed some interest in the absolute synchronization convention [6], that is on the proposal  $\phi_{\vec{v}} = (\vec{v} \cdot \vec{x})/c^2$ , where  $\vec{v}$  is the velocity of the observer  $O_{\vec{v}}$

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<sup>4</sup>Torretti [15, 16], already considered this convention for a single observer requiring that Newton’s first law be satisfied.

with respect to a privileged frame  $O_0$ . This convention, depending on the velocity  $\vec{v}$  of the observer, is not invariant under change of inertial frame. As a consequence, the laws of physics are not invariant either and the modified Lorentz transformations do not form a group. However, the Galilean relativity principle (\*) remains true because  $\phi_{\vec{v}}$  is linear in  $\vec{x}$ . Implicitly, in the previous section, much in the spirit of Mansouri and Sexl [6], a problem raised by Selleri in his paper [18] is solved, that of finding how the laws of physics must be written in the absolute synchronization convention and if there is an experiment capable of ruling it out [22]. Nevertheless, we stress, in contrast to him [19], that the possibility of anisotropic conventions does not imply the inconsistency of special relativity.

The third point requires a wider discussion; we devote the following section to it.

## 6 THE EXISTENCE OF A GLOBAL TIME VARIABLE

So far, we have considered only the case in which the  $L/c$  law holds everywhere; to generalize our treatment the field  $\vec{A}(\vec{x})$  is now taken arbitrarily, that is, we remove the condition  $\vec{A} = \nabla\phi(\vec{x})$ . If light has the velocity

$$\vec{v} = \frac{c\hat{v}}{1 + c\hat{v} \cdot \vec{A}(\vec{x})}, \quad (16)$$

then the time taken by a light beam to travel round trip over the path  $\gamma$  is given by

$$\tau = L/c + \oint_{\gamma} \vec{A} \cdot d\vec{l}, \quad (17)$$

and the difference from the case in which light travels in the opposite sense is

$$\delta\tau = 2 \oint_{\gamma} \vec{A} \cdot d\vec{l}. \quad (18)$$

This is a generalized Sagnac effect due to the distribution of velocities, Eq. (16). Being a measurable quantity, if a Sagnac effect is present, every allowed convention must account for it. Hence, in the presence of a Sagnac effect, it is impossible to find a new global time variable which allows the speed of light

to be everywhere  $c$ . The existence of a global time variable is very useful, and must be considered one of the main tasks of a good convention, so this excludes the isotropic convention in a number of physical situations. Moreover, if two conventions on the velocity of light are allowed, because they lead to the same correct predictions (Sagnac effect), then their fields are linked to one another by the gauge transformation

$$\vec{A}(\vec{x}) \longrightarrow \vec{A}(\vec{x}) + \nabla\phi(\vec{x}). \quad (19)$$

In order to see this, take the difference of two  $\vec{A}$  fields, then obtain a new field that is rotation-free because its integral over an arbitrary closed path is zero. Measurable quantities, being convention-free quantities, must be gauge invariant. For instance, the rotation  $\vec{B}(\vec{x}) = \nabla \times \vec{A}$  is gauge independent and plays the role of the field strength of the gauge theory. It can be measured revealing the Sagnac effect for a closed path in the neighborhood of  $\vec{x}$ . Again, the gauge transformation (19) follows from a resynchronization of clocks, that is from a transformation in the global time coordinate  $t \rightarrow t + \phi(\vec{x})$ .

Our treatment generalizes Anderson and Stedman's [7] who presented the choice of simultaneity, within an inertial frame, as a gauge choice.

In our formalism, the usual non-relativistic Sagnac effect [23, 24] is obtained once one takes

$$\vec{A}(\vec{x}) = \frac{\vec{\omega} \times \vec{x}}{c^2}, \quad (20)$$

where  $\omega$  is the angular velocity of the rotating platform. Indeed, with this choice, one recovers the well known formula

$$\delta\tau = \frac{4S\omega}{c^2}, \quad (21)$$

where  $S$  is the area of the surface subtended by the curve  $\gamma$ . Eq. (20) is even the best convention that people living on the Earth surface can take, where  $\vec{x}$  is the displacement from the Earth's axis.

Eq. (20) can be recovered from the general relativistic work of Møller [25] who proved that an expression like (16) with<sup>5</sup>  $A_i = g_{0i}$  holds in whatever stationary frame in which  $g_{00} = -1$ , where  $x^i = \text{const.}$  is the worldline of a generic observer at rest in the frame. In our case [25],  $g_{00} = -(1 - \frac{r^2\omega^2}{c^2})$ , and the previous condition holds, for a given radius, only in the non-relativistic limit. In more general relativistic circumstances,  $g_{00} \neq -1$ , the chosen global time variable no longer measures the rate of clocks at rest, and Eq. (16) no

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<sup>5</sup>Indices are lowered and raised with the spatial metric  $h_{ij} = g_{ij} - g_{0i}g_{0j}/g_{00}$ .

longer holds. However, using fiber bundle techniques one can still reveal the gauge nature of simultaneity, as we shall show in a forthcoming paper.

In the present paper we are mainly concerned with the case of a Euclidean space in which the  $L/c$  law holds, at least locally. In our theorem we proved that  $\vec{A} = \nabla\phi$ . However, if the  $L/c$  law does not hold globally, we can only conclude that  $\nabla \times \vec{A} = 0$ . In a space not simply connected this does not leave out the possibility of a Sagnac effect, and, as a consequence, this proves that the isotropic convention can be unsuitable even when the  $L/c$  law locally holds.

Think, as an example, of a large cylindrical spaceship of radius  $R$  spinning along its axis at angular velocity  $\omega$ . People live on the internal cylindrical surface of the spaceship and  $\omega$  is chosen to reproduce the gravitational acceleration  $g = \omega^2 R$ . Light and electric signals propagate along the surface. In such a situation the  $L/c$  law holds locally but a Sagnac effect is present when a light beam travels all around the spaceship.

## 7 CONCLUSIONS

In the first part of the paper we developed the consequences of the  $L/c$  law. We found that the allowed distributions for the velocity of light are given by Eq. (3), and that each of them can be recovered from the isotropic value via a time coordinate transformation. The relation with a coordinate transformation enabled us to rewrite the laws of physics coherently with the anisotropic convention adopted. This change in their expression clearly does not alter the physical content of the laws, so that a physics based on anisotropic conventions appears to be feasible. Moreover, anisotropic conventions can prove extremely useful, as we showed in the last part of the paper. The approach to the conventionality of simultaneity as a gauge theory seems very attractive and will be the subject of subsequent works. It can be considered a step towards general relativity.

Our analysis of the  $L/c$  was proved useful even in the causal approach to the conventionality of simultaneity. We showed that, if the  $L/c$  law holds in a Euclidean space, then the associated spacetime is causally the same as Minkowski spacetime. This result relates convention-free concepts, namely the  $L/c$  law and the causal structure of spacetime. It justifies Malament's argument if one wants to base the simultaneity concept on the causal structure. This last approach, however, appears untenable when one skips from Minkowski spacetime to more realistic spacetimes where causal automorphisms are absent.

## ACKNOWLEDGMENT

I wish to thank Bruno Carazza for his kind suggestions and for having shared useful discussions.

# APPENDIX

The theorem requires a lemma.

**Lemma.** Under the hypothesis of the theorem, let  $O_1$  be an observer at rest. Let  $t_1$  be the time measured by the clock of  $O_1$  along its own worldline  $\vec{x}_{O_1} = \text{const}$ . From the hypothesis, on the worldline of the observer, we have  $t_1 = t + \text{const}$ . where  $t$  is the global time. Outside the worldline of the observer  $O_1$  the parameter  $t_1$  is not yet defined. If  $O_1$  labels events outside his worldline using the Einstein synchronization convention, then the variable  $t_1$  becomes a global function:  $t_1 = \phi_1(\vec{x}, t)$ . Moreover,  $\phi_1$  turns out to be increasing and continuous in  $t$ . For a given  $\vec{x}$ ,  $t_1$  takes any real value and the speed of light, in this new variable, becomes  $c$ :  $d\vec{x}/dt_1 = c\hat{v}$ .

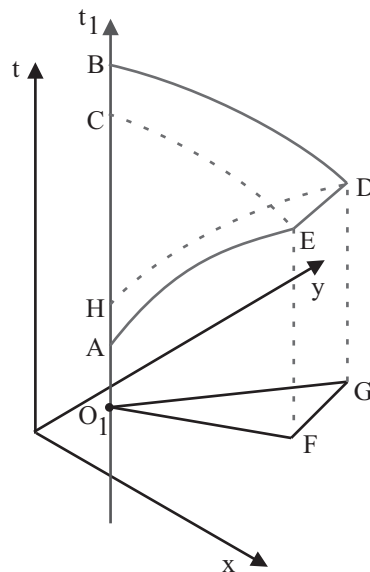


Figure 3: Note: the worldline of the light beam is not necessarily straight.

**Proof of the lemma**<sup>6</sup>. Look at figure 3 where the coordinate  $z$  is omitted for the sake of clarity. Starting from  $O_1$ , and being reflected in  $F$  and  $G$ , a light beam travels all along the path  $O_1FGO_1$ .  $FG$  is an infinitesimal displacement and our task is to show that the speed of light over  $FG$  is  $c$  if the time used is  $t_1$ . We added to the picture the trajectories of the light rays  $CE$  and  $HD$  useful to define  $t_1(D)$  and  $t_1(E)$ . From our hypothesis

$$c(t_B - t_A) = \overline{FO_1} + \overline{FG} + \overline{O_1G} \quad (22)$$

$$c(t_C - t_A) = 2\overline{FO_1} \quad (23)$$

$$c(t_B - t_H) = 2\overline{O_1G} \quad (24)$$

$$\Rightarrow \overline{FG} = \frac{c}{2}(t_B + t_H - t_A - t_C) = c(t_1(D) - t_1(E)). \quad (25)$$

Or, more explicitly

$$\frac{dt_1}{dl} = \frac{1}{c}. \quad (26)$$

For a given  $\vec{x}$ ,  $t_1 = \phi_1(\vec{x}, t)$  takes any real value. To see this, integrate the previous equation over closed paths of arbitrary length starting from  $\vec{x}$ .

The definition of  $\phi_1$  implies that  $t_1$  increases with  $t$  (because, by hypothesis, two worldlines with the same direction cannot intersect); hence  $\phi_1(\vec{x}, t)$  is invertible and  $t_1$  can be used to label events. Finally  $\phi_1$  is continuous in  $t$  because, for a given  $\vec{x}$ , it is increasing in  $t$  and its image is  $\mathbb{R}$ . The lemma is proved.

Notice that we have not yet shown that the "time" function  $t_1 = \phi_1(\vec{x}, t)$  measures the flow of time for observers different from  $O_1$ . We are ready to prove the theorem.

**Proof of the theorem.** Let us prove that a global time variable  $\eta$  which makes the speed of light isotropic must be unique up to an additive constant. Let  $\gamma$  be the worldline of a light beam of direction  $\hat{v}$ . If  $\eta_1$  makes the speed of light isotropic

$$\frac{d\eta_1}{dl} = \frac{1}{c}, \quad (27)$$

where  $l$  is the natural parameter of the projection of  $\gamma$  on  $E^3$ . Subtracting this equation with the same equation for  $\eta_2$  we find that  $\eta_1 = \eta_2 + c(\gamma)$  for all the events that lie on the worldline  $\gamma$  of the light beam considered. The constants of two different light beams must be equal if their worldlines intersect. In the coordinates  $\{\vec{x}, \eta_1\}$  the light cone of an event  $a = (\vec{x}(a), \eta_1(a))$  has the usual

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<sup>6</sup>This lemma can be generalized in the case of curved spaces. You have simply to read  $\overline{AB}$  as the length of the trajectory of the light beam from  $A$  to  $B$ .

equation  $|\vec{x} - \vec{x}(a)| = |\eta_1 - \eta_1(a)|$ , therefore the light cones of two events  $a$  and  $b$ , intersects. This implies that a light worldline passing through  $a$  intersects a light worldline passing through  $b$  and hence that

$$\eta_1(a) - \eta_2(a) = \eta_1(b) - \eta_2(b) \quad (28)$$

Making  $b$  arbitrary, we see that a constant  $c$  exists such that  $\eta_1 = \eta_2 + c$ .

Let  $t_1 = \phi_1(\vec{x}, t)$  and  $t_2 = \phi_2(\vec{x}, t)$  be the functions of the lemma related to the observers  $O_1$  and  $O_2$ . These functions are global time variables which make the speed of light isotropic. From our previous result  $t_1$  and  $t_2$  differ by an additive constant. But  $t_2$ , over the worldline  $\vec{x} = \vec{x}_{O_2}$ , equals the global time variable  $t$  up to an additive constant and hence, because  $O_2$  is arbitrary,

$$\phi_1(\vec{x}, t) = t - \tilde{\phi}_1(\vec{x}) \quad (29)$$

for a suitable scalar function  $\tilde{\phi}_1(\vec{x})$ . This proves a).

Let now  $f$  be continuous in its arguments. By hypothesis the worldline  $\vec{x}(t)$  of a light beam of direction  $\hat{v}$  is derivable (see Eq. (4)). This implies that  $t(l)$  is derivable in the open set  $A \subset S^2 \times M$  where  $f(\hat{v}, \vec{x}, t) > 0$ . The lemma shows that  $t_1(l)$  is derivable too, therefore the same is true for  $\tilde{\phi}_1(\vec{x}(l))$  and we obtain

$$\nabla_{\hat{v}} \tilde{\phi}_1(\vec{x}) = \frac{1}{f(\hat{v}, \vec{x}, t)} - \frac{1}{c} \quad \text{in } A. \quad (30)$$

This proves that  $f$  is independent of time in  $A$ . Considering only the  $t$  dependence,  $f$  takes two values: zero outside  $A$  and  $c/[1 + c\nabla_{\hat{v}} \tilde{\phi}_1(\vec{x})]$  inside  $A$ . However,  $f$  is continuous in  $t$  and hence independent of time throughout  $S^2 \times M$ .

If  $f(\hat{v}, \vec{x}) > 0$  there is a neighborhood of  $\hat{v} \in S^2$  where we can find three linear independent vectors  $\hat{v}_1, \hat{v}_2, \hat{v}_3$  which verify  $f(\hat{v}_i, \vec{x}) > 0$ . The partial derivatives  $\nabla_{\hat{v}_i} \tilde{\phi}_1(\vec{x})$  are continuous in  $\vec{x}$  (see Eq. (30)) and therefore  $\tilde{\phi}_1$  is differentiable:  $\nabla_{\hat{v}} \tilde{\phi}_1(\vec{x}) = \hat{v} \cdot \nabla \tilde{\phi}_1(\vec{x})$ .

Let  $B \subset S^2 \times E^3$  be the open subset where  $f(\hat{v}, \vec{x}) > 0$ . From Eq. (30),

$$f(\hat{v}, \vec{x}) = \frac{c}{1 + c\hat{v} \cdot \nabla \tilde{\phi}_1(\vec{x})} \quad \text{in } B, \quad (31)$$

therefore  $f \geq c/[1 + c|\nabla \tilde{\phi}_1(\vec{x})|]$  in  $B$ . If  $(\hat{v}_1, \vec{x}_1) \in B$  then  $\forall \hat{v} (\hat{v}, \vec{x}_1) \in B$ , indeed, for a given  $\vec{x}_1$ , there is no continuous function  $f : S^2 \rightarrow \mathbb{R}$  which is greater than  $c/[1 + c|\nabla \tilde{\phi}_1(\vec{x}_1)|]$  in an open subset of  $S^2$  and zero elsewhere. Let  $C \subset E^3$  be the open subset where  $f(\hat{v}, \vec{x}) > 0$ . From Eq. (31), because  $f$  is finite,  $|\nabla \tilde{\phi}_1(\vec{x})| < 1/c$  or  $f(\hat{v}, \vec{x}) > c/2$  in  $C$ . But  $f$  is continuous in  $\vec{x}$



throughout  $E^3$  therefore  $C = E^3$ ,  $B = S^2 \times E^3$  and  $A = S^2 \times M$ . In other words  $f$  is positive. Eq. (31) proves statement b).

In order to prove c) notice that the causal structure does not depend on the coordinates used to label events. From a) and using the coordinates  $\{\vec{x}, t_1\}$  we recover the light cone structure of Minkowski spacetime, and since, by hypothesis, light is the fastest signal, the causal structure of  $M$  coincides with that of Minkowski spacetime.

## References

- [1] H. Poincaré, *Rev. Metaphys. Morale* **6**, (1898)
- [2] A. Einstein, *Relativity, The Special and General Theory: A Popular Exposition*, (Methuen, London, 1920)
- [3] H. Reichenbach, *The Philosophy of Space and Time*, (Dover, New York, 1958)
- [4] A. Grünbaum, *Philosophical Problems of Space and Time*, (Reidel Publishing Company, Dordrecht/Boston, 1973)
- [5] A. S. Eddington, *The Mathematical Theory of Relativity*, (Cambridge University Press, Cambridge, 1924)
- [6] R. Mansouri and R. Sexl, *Gen. Rel. Gravit.* **8**, 497 (1977)
- [7] R. Anderson and G.E. Stedman, *Found. Phys. Lett.* **5**, 199 (1992)
- [8] R. Anderson, I. Vetharaniam, and G.E. Stedman, *Phys. Rep.* **295**, 93 (1998)
- [9] M. Jammer, “Some fundamental problems in the special theory of relativity” in *Problems in the Foundations of Physics*. G. Toraldo di Francia ed. (North-Holland, Amsterdam, 1979)
- [10] C. M. Will, *Phys. Rev. D* **45**, 403 (1992)
- [11] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, (Cambridge University Press, Cambridge, 1973) p. 60
- [12] D. Malament, *Noûs* **11**, 293 (1977)
- [13] T. Budden, *Found. Phys. Lett.* **11**, 343 (1998)

- [14] M. Vallisneri *Relativity and Acceleration*, Ph.D. thesis, Università di Parma (2000)
- [15] R. Torretti, *Relativity and Geometry*, (Dover, New York, 1993)
- [16] R. Torretti, *The Philosophy of Physics*, (Cambridge University Press, Cambridge, 1999)
- [17] A. Ungar, *Phil. Sci.* **53**, 395 (1986)
- [18] F. Selleri, *Found. Phys.* **26**, 641 (1996)
- [19] F. Selleri, *Found. Phys. Lett.* **10**, 73 (1997)
- [20] S. Bergia and M. Guidone *Found. Phys. Lett.* **11**, 549 (1998)
- [21] G. Rizzi and A. Tartaglia, *Found. Phys. Lett.* **12**, 179 (1999)
- [22] J. R. Crocca and F. Selleri, *Nuovo Cimento* **114 B**, 447 (1999)
- [23] M. G. Sagnac, *C. R. Acad. Sci.* **157**, 708 (1913); *ibid.* 1410
- [24] G. E. Stedman, *Rep. Prog. Phys.* **60**, 615 (1997) and references therein.
- [25] C. Møller, *The Theory of Relativity*, (Claredon, Delhi, 1972)